Universiteit Utrecht

A new logic of natural relations

CS Colloquium
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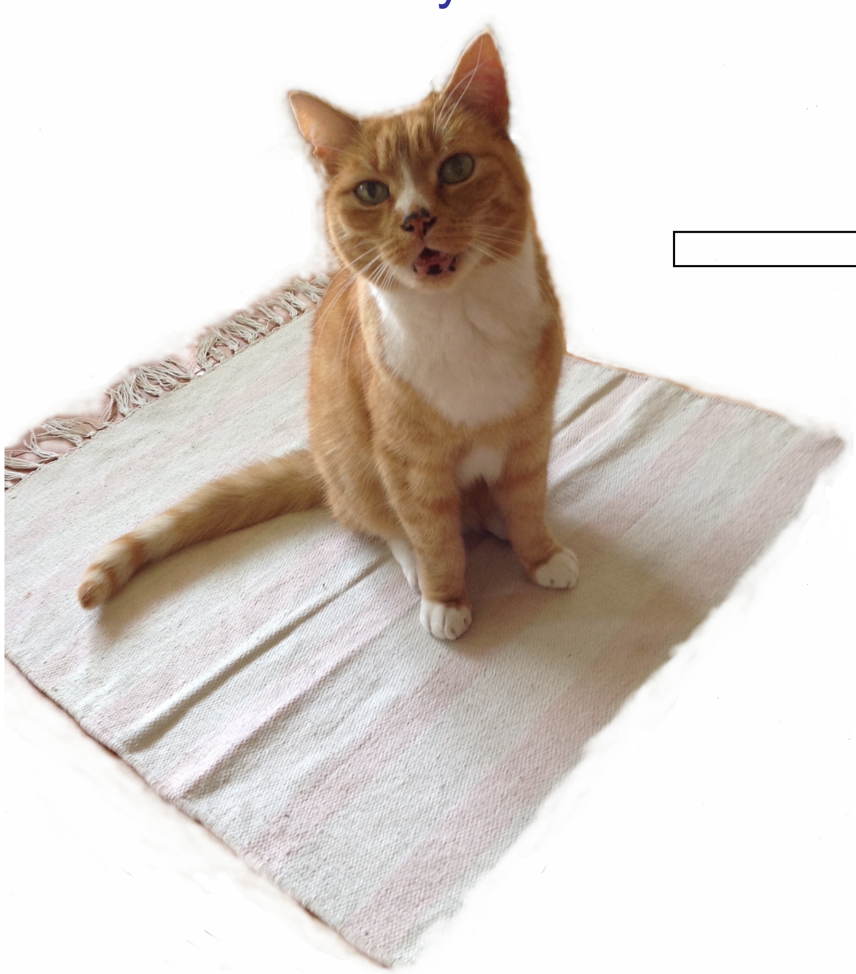
Outline

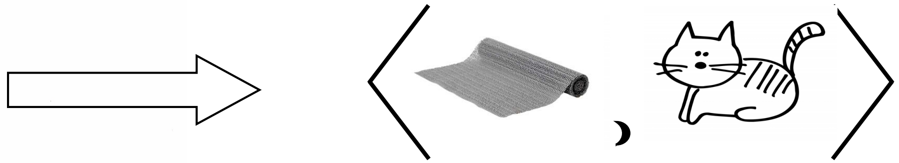
- Metaphysical views on relations
- Why we need a new logic
- Developing a minimalistic logic
- Developing a logic of relations
- Impact

Standard view on relations

Reality



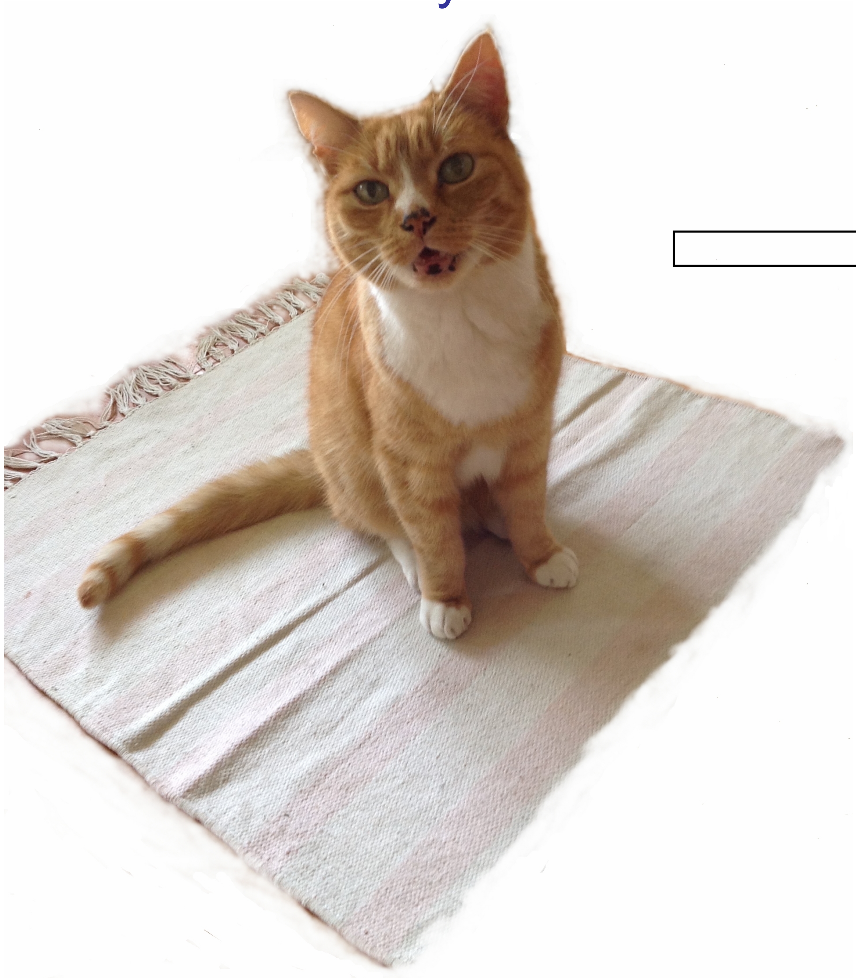
on top of
beneath



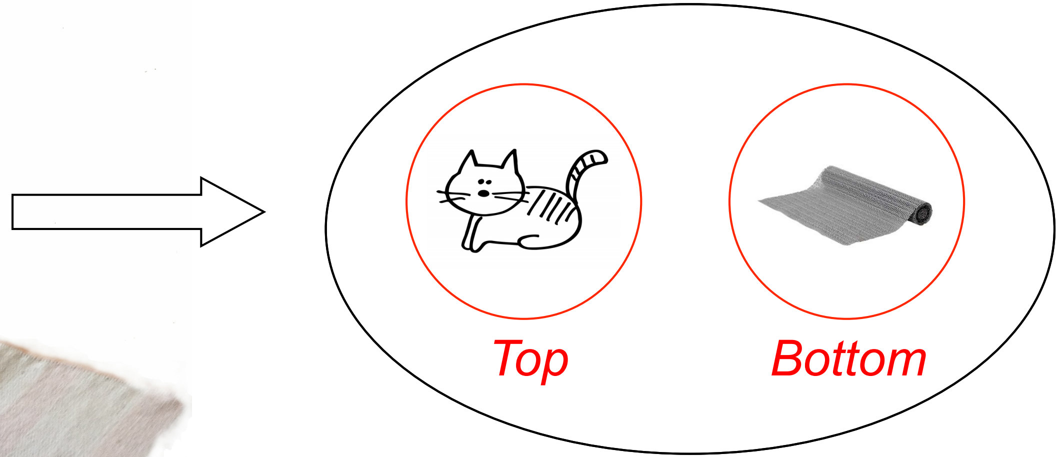
The arguments come in a certain order

Positionalist view on relations

Reality



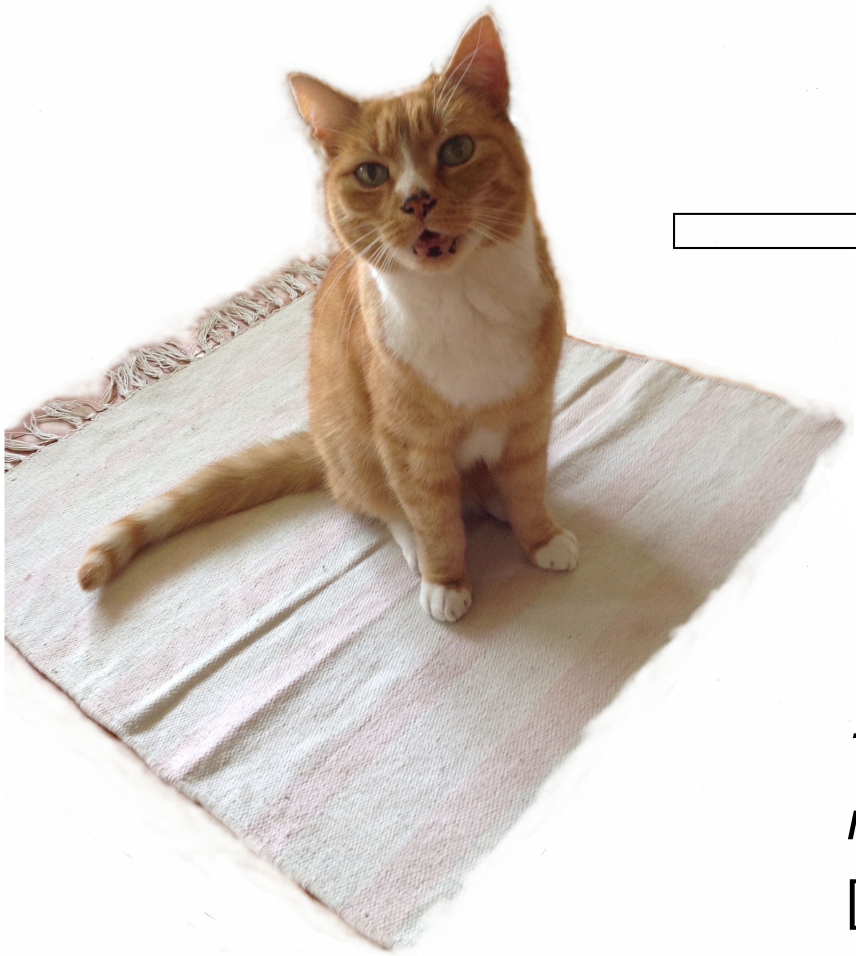
vertical placement

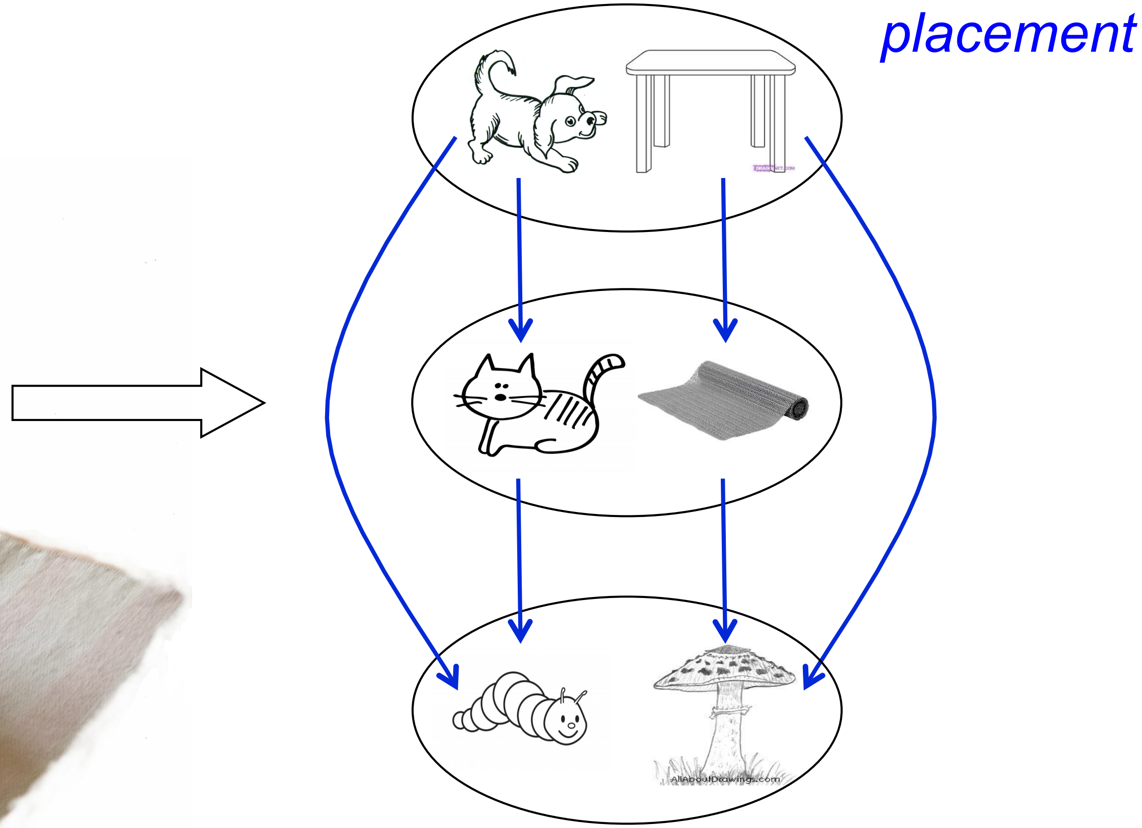


But what about symmetric relations?

Antipositionalist view on relations

Reality

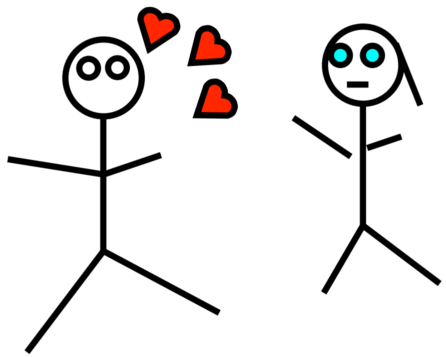




The complexes of a relation form a network interrelated by substitutions

[Fine 2000, Leo 2010]

Existing logics



Adam loves Eve

Adam loves Eve

loves (Adam, Eve)

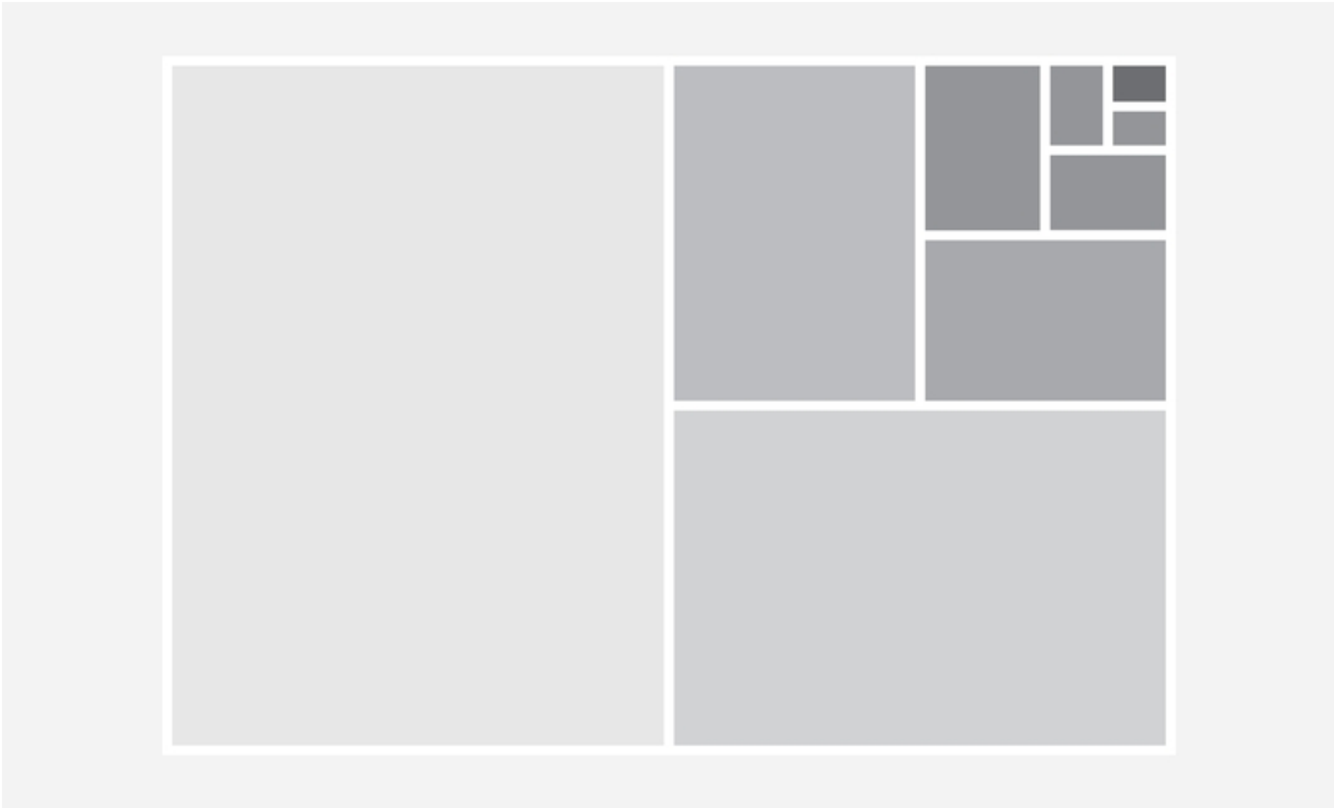
They function like distorting mirrors

Why we need a new logic

1. Antipositionalism is the superior view on relations

2. Existing logics do not correspond to antipositionalism

3. We expect of an impeccable logic that it can represent reality in a very natural way



Developing a minimalistic logic

Take a fresh look at the world

We look at the world as consisting of entities with input–output behavior

Minimalistic logic

Example

Terms tomato, orange, blender, tomato juice, x, …

tomato(tomato), blender(tomato), …

Formulas tomato = tomato

tomato(tomato) = tomato(tomato)

blender(tomato) = tomato juice

$\neg$(blender(orange) = tomato juice)

$\forall$x (blender(x) = tomato juice $\rightarrow$ x = tomato)

Minimalistic logic

Symbols:	simple terms	*a*, *b*, *x*, …
	equality symbol	=
	application symbol	$\cdot(\cdot)$
	connectives	$\wedge$, $\neg$, $\forall$

Terms: simple terms

for all terms t, t', the term $t(t')$

Formulas: for all terms t, t', the formula $t = t'$

for all formulas φ, ψ, the formulas $(\varphi \wedge \psi)$, $\neg\varphi$

for all formulas φ and simple terms x, the formula $\forall x\ \varphi$

Minimalistic logic

A *minimalistic structure* is a (possibly empty) collection E of entities that may have input–output functionality with all inputs and outputs belonging to E as well.

If x is an input of e, then we denote the output as $e(x)$.

- We allow different entities to have the same input–output functionality.
- We allow entities to have themselves as inputs.

Minimalistic logic

Minimalistic structures

Set-theoretic structures

Minimalistic logic

Semantics

Let E be a minimalistic structure.

Let g: simple terms $\to E$ be a partial function.

Interpreting terms

- $\lceil t \rceil_{E,g} = g(t)$ if t is a simple term
- $\lceil t(t') \rceil_{E,g} = \lceil t \rceil_{E,g}(\lceil t' \rceil_{E,g})$

Note that $\lceil t \rceil_{E,g}$ may be undefined.

Minimalistic logic

Interpreting formulas

Define $V_{E,g}$: Formulas $\rightarrow \{0, 1\}$ as a total function such that

- $V_{E,g}(t = t') = 1$ iff $\lceil t \rceil_{E,g}$ and $\lceil t' \rceil_{E,g}$ are defined and $\lceil t \rceil_{E,g} = \lceil t' \rceil_{E,g}$
- $V_{E,g}(\varphi \wedge \psi) = 1$ iff $V_{E,g}(\varphi) = 1$ and $V_{E,g}(\varphi) = 1$
- $V_{E,g}(\neg\varphi) = 1$ iff $V_{E,g}(\varphi) = 0$
- $V_{E,g}(\forall x\, \varphi) = 1$ iff $V_{E,g[x\,:\,e]}(\varphi) = 1$ for every $e \in E$

where $g[x : e]$ maps x to e, and any other simple term y to $g(y)$

Minimalistic logic

Existence predicate

$\mathrm{E!}\ t \quad =_{df} \quad \exists x\,(x = t)$, with x not in t

Weak equality

$t \simeq t' \quad =_{df} \quad \mathrm{E!}\ t \vee \mathrm{E!}\ t' \rightarrow t = t'$

Minimalistic logic

Axioms: Tautologies

$\forall x\,(\varphi \to \psi) \to (\forall x\,\varphi \to \forall x\,\psi)$

$\varphi \to \forall x\,\varphi$, where x does not occur free in φ

$\forall x\,\varphi(x) \to (\mathrm{E!}\,t \to \varphi(t))$, where t is substitutable for x in φ

$\forall x\; x = x$

$t = t' \to (\varphi \leftrightarrow \varphi')$, where φ' is obtained from φ by zero or more substitutions of t' for t where both t and t' occur free

$t = t \to \mathrm{E!}\,t$

$\mathrm{E!}\,t(t') \to \mathrm{E!}\,t \wedge \mathrm{E!}\,t'$

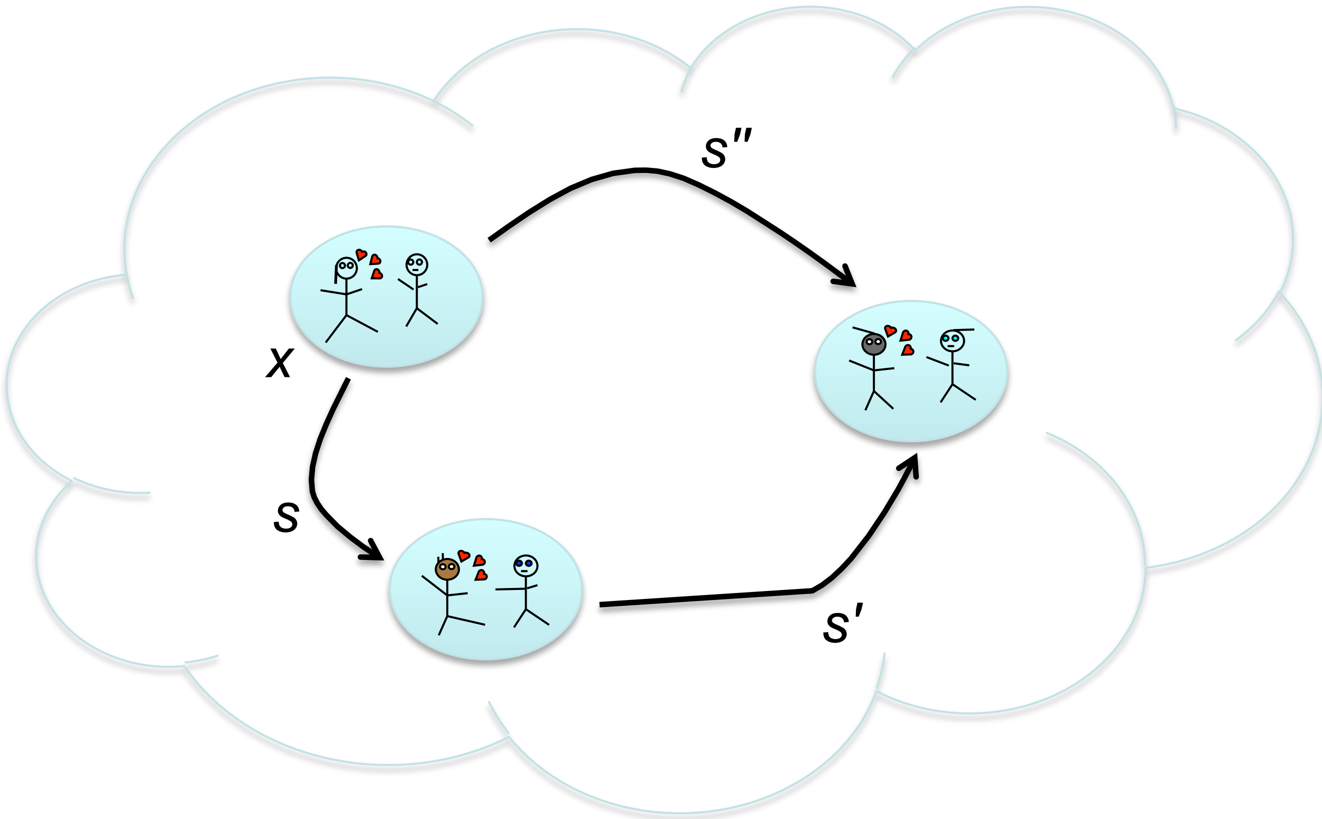
Rule of inference: from φ and $\varphi \to \psi$ infer ψ

Minimalistic logic

Minimalistic logic is
conceptually simpler than predicate logic,
but its proof-theoretic strength is the same.

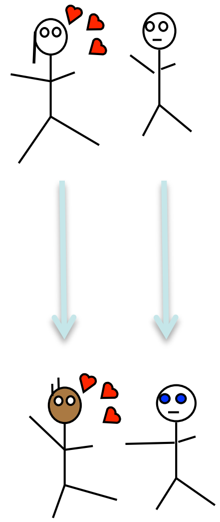
How does the love relation fit within this minimalistic framework?

Developing a logic of relations



The complexes of a relation form a network interrelated by substitutions

Developing a logic of relations



We call a term *x* a *complex* if it fulfills the following axioms:

1. Identity substitution:

$$\exists! s\ (x(s) = x \wedge \forall \alpha\ (E!\ s(\alpha) \rightarrow s(\alpha) = \alpha))$$

2. Composition of substitutions:

$$E!\ x(s)(s') \ \rightarrow\ \exists! s''\ (x(s)(s') = x(s'') \wedge \forall \alpha\ (s'(s(\alpha)) \simeq s''(\alpha)))$$

…

Developing a logic of relations

Example: The love relation (simplified)

Axioms: E! a_loving_b → a_loving_b is a complex

E! a_loving_b → E! adam ∧ E! eve ∧ adam ≠ eve

E! a_loving_b → ∀*x* (*x* in a_loving_e ↔ *x* = adam ∨ *x* = eve)

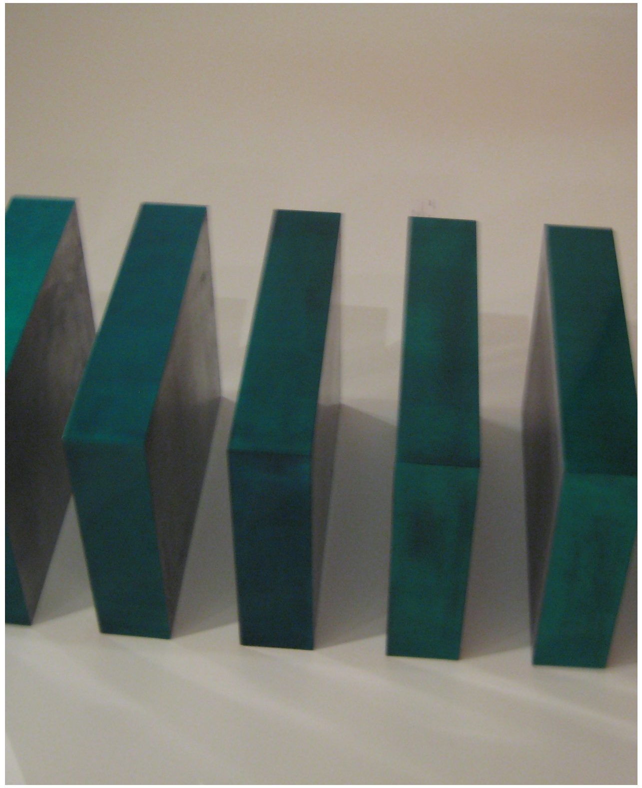
where *α* in *x* $=_{df}$ ∃*s* (E! x(s) ∧ E! *s*(*α*))

How do we express that Romeo loves Juliet?

∃*s* (E! a_loving_b(*s*) ∧ *s*(adam) = romeo ∧ s(eve) = juliet)

Impact

Facilitates
coordinate-free thinking



Drives development of a
new foundation of mathematics

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